

Dirac's reduction of linearized gravity in $N > 2$ dimensions revisited

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We perform a brief review on Dirac's procedure applied to the well known Einstein's linearized gravity in $N > 2$ dimensions. Considering it as a gauge theory and therefore the manifestation of second class constraints in analogy with the electromagnetic case, focussing our interest in the Coulomb's gauge. We also check the consistency with the Maskawa-Nakajima reduction procedure and end with some remarks on both procedures.

I. INTRODUCTION

For a long time there has been great interest on the procedure to quantize gravity in a perturbative regime using known techniques coming from Quantum Field Theory. In the Fock's space formalism, for example, it is confirmed that graviton's spin is 2 thanks to the representation of the Poincaré algebra via the creation-annihilation operators[1]. Later, a series of studies began in the context of Hamiltonian formalism with the goal of reducing the degrees of freedom of Einstein's gravity by the imposition of constraints that don't change the equations of motion, even though they change the Lagrangian density and we must abandon covariance[2].

There are other perspectives to engage the degrees of freedom reduction of gravity in the Hamiltonian formalism and the Dirac's constraints analysis. The light-cone[3] or the null-plane[4] coordinates fixings which not necessarily means a priori gauge fixing, the light-cone gauge fixing [5] are just a few of them.

The main purpose of our work is to aboard the Dirac's analysis for Einstein's linearized gravity by making some analogies with Maxwell's theory as the analogous Coulomb's gauge fixing for gravity. A comparison with the Maskawa-Nakajima[6] procedure is also discussed. As usual, we'll decompose tensors of rank 1 and 2 following the known methodology[7].

This paper is organized as follows. In the next section we study Dirac's procedure for Einstein's Linearized Gravity with the Coulomb's gauge. Then, we perform the Maskawa-Nakajima reduction to compare Maxwell's theory with Linearized Gravity in the previously mentioned gauge, where the projectors of spin 1 and 2 merge naturally. Finally, we end with some comments.

II. NOTATION

The Hilbert-Einstein action, S_{HE} describes gravity under the postulates of general relativity and it comes as

$$S_{HE} = -\frac{1}{k^{N-2}} \int d^{N-1}x \sqrt{-g} \mathcal{R}, \quad (1)$$

where g is the determinant of the metric tensor, k is a proportionality constant which comes in units of length and \mathcal{R} is the Ricci's scalar, defined as

$$\mathcal{R} = g^{\mu\nu} \mathcal{R}_{\mu\nu} = g^{\mu\nu} \mathcal{R}^\lambda_{\mu\lambda\nu}, \quad (2)$$

where $\mathcal{R}_{\mu\nu}$ are the components of Ricci's tensor and $\mathcal{R}^\lambda_{\mu\sigma\nu}$ the components of Riemann-Christoffel's tensor. Here we think on a N -dimensional ($N > 2$) space-time with null metricity ($\nabla_\alpha g_{\mu\nu} = 0$) and torsionless ($T^\alpha_{\mu\nu} = 0$) therefore, the Levi-Civita's connection comes from Christoffel's symbols as $\Gamma^\mu_{\lambda\nu}$ only in terms of the components of the metric tensor and its first derivatives in the usual way $\Gamma^\mu_{\lambda\nu} = \frac{g^{\mu\rho}}{2} (\partial_\lambda g_{\rho\nu} + \partial_\nu g_{\rho\lambda} - \partial_\rho g_{\lambda\nu})$.

So, the components of Riemann-Christoffel's tensor comes as

$$\mathcal{R}^\alpha_{\mu\nu\lambda} = \partial_\lambda \Gamma^\alpha_{\nu\mu} - \partial_\nu \Gamma^\alpha_{\lambda\mu} + \Gamma^\alpha_{\lambda\sigma} \Gamma^\sigma_{\nu\mu} - \Gamma^\alpha_{\nu\sigma} \Gamma^\sigma_{\lambda\mu}, \quad (3)$$

establishing that Ricci's scalar in the action (1) have a dependence until second order in derivatives of the components of the metric.

When one perform arbitrary functional variations on the metric at the Hilbert-Einstein's action, it can be shown that it's an extremal if

$$G_{\alpha\beta} \equiv \mathcal{R}_{\alpha\beta} - \frac{g_{\alpha\beta}}{2} \mathcal{R} = 0, \quad (4)$$

where $G_{\alpha\beta}$ is the Einstein's tensor. With all these, the perturbative analysis is thought in the surrounding stationary points of the action S_{HE} .

First order perturbations in the metric are made in a usual way around a Minkowski's background, this means

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad (5)$$

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}, \quad (6)$$

where $\eta = \text{diag}(-1, 1, \dots, 1)$ is the Minkowski's metric tensor and $h_{\mu\nu} \ll 1$ is the perturbation. At first order, we rise and down indexes with Minkowski's metric.

Now, we can write the linearized version of (1) in terms of the field $h_{\mu\nu}$ as follows

$$S^L = -\frac{1}{k^{N-2}} \int d^{N-1}x h_{\mu\nu} G^{L\mu\nu}(h), \quad (7)$$

where

$$\begin{aligned} G_{\mu\nu}^L(h) &= \mathcal{R}_{\mu\nu}^L - \frac{\eta_{\mu\nu}}{2} \mathcal{R}^L \\ &= \frac{1}{2} [\Box h_{\mu\nu} + \partial_\mu \partial_\nu h - \partial_\mu \partial_\alpha h_\nu^\alpha \\ &\quad - \partial_\nu \partial_\alpha h_\mu^\alpha + \eta_{\mu\nu} (\Box h - \partial_\alpha \partial_\beta h^{\alpha\beta})] . \end{aligned} \quad (8)$$

In this context, the perturbative field $h_{\mu\nu}$ is a rank 2 tensor that transforms under the (locally) Lorentz group and, due to the diffeomorphism's symmetry, it also has the well known functional transformation

$$\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu , \quad (9)$$

where ξ_μ is an arbitrary vector field, which is continuous and differentiable, recalling us in some way to the form in which the vector potential of Maxwell's theory transforms due to the gradient of a scalar field. It's very common to find references of this diffeomorphism transformation as a gauge invariance that can be found formally using Castellani's procedure [8]. In this sense, we'll use this to make the transition to the second class constraints system, i.e. fixing a gauge in linearized gravity having in mind the analogy with the fixing of Coulomb's gauge.

III. DIRAC'S PROCEDURE IN THE PERTURBATIVE REGIME

We begin by doing a $(N-1)+1$ decomposition of the linearized action (7) that we have redefined as $S^{*L} \equiv -k^{N-2} S^L$, so it follows

$$\begin{aligned} S^{*L} &= \left\langle \frac{1}{2} \dot{h}_{ij} \dot{h}_{ij} - \frac{1}{2} \left(\dot{h}_{kk} \right)^2 + 2 \left(\dot{h}_{kk} \partial_l h_{0l} - \dot{h}_{kl} \partial_k h_{0l} \right) \right. \\ &\quad + h_{0l} \left(-\Delta h_{0l} + \partial_l \partial_k h_{0k} \right) + h_{ij} \left(\frac{1}{2} \Delta h_{ij} - \partial_i \partial_k h_{kj} \right. \\ &\quad \left. \left. + \partial_i \partial_j h_{kk} - \frac{\eta_{ij}}{2} \Delta h_{kk} \right) + h_{00} \left(\Delta h_{kk} - \partial_l \partial_k h_{kl} \right) \right\rangle . \end{aligned} \quad (10)$$

Following the Dirac's procedure, we write the canonical momentum $\Pi^{\alpha\beta} \equiv \frac{\partial L}{\partial h_{\alpha\beta}}$, to get

$$\Pi^{0\mu} = 0 , \quad (11)$$

$$\Pi^{ij} = \dot{h}_{ij} - \eta_{ij} \dot{h}_{kk} + 2\eta_{ij} \partial_l h_{0l} - \partial_i h_{0j} - \partial_j h_{0i} . \quad (12)$$

We notice that (11) is a primary constraint $\phi_1^\mu \equiv \Pi^{0\mu}$, while (12) is an expression that allow us to find the velocities \dot{h}_{ij} . With these, we can write the Hamiltonian

density H_0 of the system

$$\begin{aligned} H_0 &= \frac{\Pi_{ij} \Pi_{ij}}{2} - \frac{(\Pi_{kk})^2}{2(N-2)} - h_{ij} \left(\frac{\Delta h_{ij}}{2} - \eta_{ij} \Delta h_{kk} \right. \\ &\quad \left. - \partial_i \partial_k h_{kj} - \partial_j \partial_k h_{ki} + \partial_i \partial_j h_{kk} \right) + h_{00} \left(\partial_l \partial_k h_{kl} \right. \\ &\quad \left. - \Delta h_{kk} \right) - 2h_{0j} \partial_i \Pi^{ij} , \end{aligned} \quad (13)$$

and the total Hamiltonian density can be build if we include the primary constraint with a Lagrange multiplier

$$H_T = H_0 + u_\mu \Pi^{0\mu} . \quad (14)$$

To continue with the procedure, we preserve the primary constraint ϕ_1^μ using Poisson's brackets algebra for symmetric rank 2 fields, which by construction they come as:

$$\left\{ h_{\alpha\beta}(x), \Pi^{\mu\nu}(y) \right\} = \frac{1}{2} (\delta_\alpha^\mu \delta_\beta^\nu + \delta_\alpha^\nu \delta_\beta^\mu) \delta^{N-1}(x-y) . \quad (15)$$

Hence, the preservation of the aforementioned constraint gives

$$\begin{aligned} \dot{\phi}_1^\mu(x) &= \int d^{N-1}y \left\{ \phi_1^\mu(x), H_T(y) \right\} \\ &= \delta_0^\mu (\Delta h_{kk} - \partial_l \partial_k h_{lk}) + \delta_j^\mu \partial_i \Pi^{ij} = 0 , \end{aligned} \quad (16)$$

representing N new constraints which components are $\phi_2^0 \equiv \Delta h_{kk} - \partial_l \partial_k h_{lk}$ and $\phi_2^j \equiv \partial_i \Pi^{ij}$. We must preserve these, so it can be obtained

$$\dot{\phi}_2^0 = \int d^{N-1}y \left\{ \phi_2^0(x), H_T(y) \right\} = -\partial_i \phi_2^i \equiv 0 , \quad (17)$$

$$\dot{\phi}_2^j = \int d^{N-1}y \left\{ \phi_2^j(x), H_T(y) \right\} = 0 . \quad (18)$$

No more new constraints appear, so the process of preservation ends. Therefore, we resume the constraints

$$\phi_1^\mu \equiv \Pi^{0\mu} , \quad (19)$$

$$\phi_2^0 \equiv \Delta h_{kk} - \partial_l \partial_k h_{lk} , \quad (20)$$

$$\phi_2^j \equiv \partial_i \Pi^{ij} , \quad (21)$$

and we immediately note that all of them are first class constraints. Ahead we'll extend this system to a second class one when we choose a gauge. The physical reason of this comes from the ambiguity due to the gauge freedom that lead us to the fact that not all of the fields are actually local degrees of freedom. This is confirmed by noticing that the Hamiltonian (13) is not positively

defined in analogy with Maxwell's theory with gauge freedom.

Considering then that Einstein's linearized gravity have an N parameters gauge invariance presented in (9), we choose N gauges similar to the Coulomb gauge via the following ad hoc constraints

$$\chi_\mu \equiv \partial_i h_{i\mu} , \quad (22)$$

meaning N new constraints imposed which must be preserved as the Dirac's procedure says. So the preservation of them leads to

$$\begin{aligned} \dot{\chi}_\mu &= \int d^{N-1}y \left\{ \chi_\mu(x), H_T(y) \right\} \\ &= \int d^{N-1}y \left\{ \chi_\mu(x), H_0(y) \right\} + \frac{1}{2} \delta_\mu^0 \partial_i u_i(x) . \end{aligned} \quad (23)$$

If we take $\mu = 0$, we get a differential equation for $N - 1$ Lagrange multipliers without getting any new constraints, this means

$$\partial_i u_i(x) \simeq -2 \int d^{N-1}y \left\{ \partial_i h_{i0}(x), H_0(y) \right\} . \quad (24)$$

However, if we take $\mu = j$ in (23) we get $N - 1$ new constraints

$$\chi_j^2 \equiv -\frac{1}{N-2} \partial_j \Pi_{kk} + \triangle h_{0j} , \quad (25)$$

and their preservation give

$$\dot{\chi}_j^2 = \int d^{N-1}y \left\{ \chi_j^2(x), H_0(y) \right\} + \frac{1}{2} \triangle u_j(x) \quad (26)$$

which means $N - 1$ Poisson type equations for the multipliers u_j and due to consistency with (24) and using (25) they can be written as follows

$$\begin{aligned} \partial_j \dot{\chi}_j^2 &= \partial_j \int d^{N-1}y \left\{ \chi_j^2(x), H_0(y) \right\} + \frac{1}{2} \triangle \partial_j u_j(x) \\ &= -\frac{1}{N-2} \triangle \int d^{N-1}y \left\{ \Pi_{kk}(x), H_0(y) \right\} \\ &= (N-2) \triangle^2 h_{00}(x) , \end{aligned} \quad (27)$$

which up to harmonic forms we get a new constraint

$$\chi_3 \equiv h_{00} . \quad (28)$$

Its preservation follows as

$$\dot{\chi}_3 = \int d^{N-1}y \left\{ \chi_3(x), H_T(y) \right\} = u_0(x) , \quad (29)$$

and with this we can determine the remaining Lagrange multiplier and the preservation process ends.

We rename the constraints and make a list of all of them in the following way

$$\chi_1 \equiv \Pi^{00} , \quad (30)$$

$$\chi_1^i \equiv \Pi^{0i} , \quad (31)$$

$$\chi_2 \equiv h_{kk} , \quad (32)$$

$$\chi_2^j \equiv \partial_i \Pi^{ij} , \quad (33)$$

$$\chi_3^j \equiv \partial_i h_{ij} , \quad (34)$$

$$\chi_4 \equiv \Pi_{kk} , \quad (35)$$

$$\chi_4^j \equiv h_{0j} , \quad (36)$$

$$\chi_5 \equiv h_{00} . \quad (37)$$

This means that we have a system of $4N$ second class constraints, and since there are $N(N+1)$ fields and canonically conjugate momenta, we finally have $\frac{N(N+1)-4N}{2} = \frac{N(N-3)}{2}$ degrees of freedom.

The next step is to build Dirac's matrix using the Poisson brackets of the constraints, so we get a $4N \times 4N$ range matrix that is written as

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\eta_{ij}}{2} & 0 \\ 0 & 0 & 0 & -\partial_i & 0 & (N-1) & 0 & 0 \\ 0 & 0 & \partial_j & 0 & \frac{\eta_{ij}\Delta + \partial_i \partial_j}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\eta_{ij}\Delta + \partial_i \partial_j}{2} & 0 & \partial_j & 0 & 0 \\ 0 & 0 & -(N-1) & 0 & -\partial_i & 0 & 0 & 0 \\ 0 & \frac{\eta_{ij}}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \delta_{(x-x')}^{N-1} . \quad (38)$$

To continue Dirac's procedure it is necessary to find the inverse $C^{-1}(x' - y)$, which must satisfies the property

$$\int d^{N-1}x' C(x - x') C^{-1}(x' - y) = \mathbb{I} \delta^{N-1}(x - y) , \quad (39)$$

where \mathbb{I} is the identity matrix with $4N \times 4N$ range.

We make an ansatz over the form of this inverse matrix, in a similar way to the form that the original matrix $C(x - x')$ have, to get

$$C^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta^{N-1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 2\eta_{jl}\delta^{N-1} & 0 \\ 0 & 0 & 0 & -A_j & 0 & -E & 0 & 0 \\ 0 & 0 & A_l & 0 & -B_{jl} & 0 & 0 & 0 \\ 0 & 0 & 0 & B_{jl} & 0 & -D_l & 0 & 0 \\ 0 & 0 & E & 0 & D_j & 0 & 0 & 0 \\ 0 & -2\eta_{jl}\delta^{N-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\delta^{N-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}_{(x'-y)} , \quad (40)$$

where $A_i(x' - y)$, $B_{ij}(x' - y)$, $D_i(x' - y)$ and $E(x' - y)$ are undetermined functions. From (39) it arises a set of

consistency rules

$$-\partial_i A_i(x-y) + (N-1)C(x-y) = \delta_{(x-y)}^{N-1} \quad (41)$$

$$\partial_j B_{ij}(x-y) + (N-1)D_i(x-y) = 0 \quad (42)$$

$$-\partial_j A_i(x-y) + \left(\frac{\eta_{ij} \Delta + \partial_l \partial_j}{2}\right) B_{lj}(x-y) = \eta_{ij} \delta_{(x-y)}^{N-1} \quad (43)$$

$$\partial_j C(x-y) + \left(\frac{\eta_{ij} \Delta + \partial_l \partial_j}{2}\right) D_i(x-y) = 0 \quad (44)$$

$$-\left(\frac{\eta_{ij} \Delta + \partial_l \partial_j}{2}\right) A_j(x-y) + \partial_i C(x-y) = 0 \quad (45)$$

$$\left(\frac{\eta_{ij} \Delta + \partial_l \partial_j}{2}\right) B_{il}(x-y) + \partial_j D_i(x-y) = \eta_{ij} \delta_{(x-y)}^{N-1} \quad (46)$$

$$(N-1)A_i(x-y) - \partial_j B_{ij}(x-y) = 0 \quad (47)$$

$$(N-1)C(x-y) + \partial_j D_j(x-y) = \delta_{(x-y)}^{N-1}, \quad (48)$$

which allow us to determine the unknown functions in the inverse matrix. For this task we use the solution of the $N > 2$ dimension Poisson's equation [9], so we find a solution of the system (41 - 48) as follows

$$A_i(x-y) = -D_i(x-y), \quad (49)$$

$$B_{ij}(x-y) = \left(2\eta_{ij} + \frac{(N-3)}{(N-2)}\hat{\partial}_i \hat{\partial}_j\right)\phi(x-y), \quad (50)$$

$$C(x-y) = \frac{1}{N-2}\delta^{N-1}(x-y), \quad (51)$$

$$D_j(x-y) = -\frac{1}{(N-2)}\partial_j \phi(x-y), \quad (52)$$

where

$$\phi(x-y) \equiv \begin{cases} \frac{1}{2\pi} \ln|x-y| & \text{if } N=3 \\ -\frac{1}{\alpha_{(N)}|x-y|^{N-3}} & \text{if } N \geq 4 \end{cases}, \quad (53)$$

with $\alpha_{(N)} = \frac{\Gamma(\frac{N-1}{2})}{2(N-3)\pi^{\frac{N-1}{2}}}$ denotes the surface area of a N -sphere in \mathbb{R}^N and $\hat{\partial}_i \equiv \frac{\partial_i}{\sqrt{-\Delta}}$.

With this, we build Dirac's brackets in the usual way $\{A(x'), B(y')\}_D = \{A(x'), B(y')\} - \int d^{N-1}x d^{N-1}y \{A(x'), \chi_s(x)\} C_{ss'}^{-1}(x-y) \{\chi_{s'}(y), B(y')\}$ and all the constraints are now first class, so we can take them strongly equal to zero. So the brackets that are

not zero are

$$\begin{aligned} & \{h_{ij}(x), \Pi^{kl}(y)\}_D \\ &= \left[\frac{1}{2}(\eta_{ik}\eta_{jl} + \eta_{il}\eta_{jk} + \eta_{ik}\hat{\partial}_j\hat{\partial}_l + \eta_{jl}\hat{\partial}_i\hat{\partial}_k \right. \\ &+ \eta_{il}\hat{\partial}_j\hat{\partial}_k + \eta_{jk}\hat{\partial}_i\hat{\partial}_l) - \frac{1}{N-2}(\eta_{ij}\eta_{kl} + \eta_{ij}\hat{\partial}_k\hat{\partial}_l \\ &+ \eta_{kl}\hat{\partial}_i\hat{\partial}_j) + \left. \frac{N-3}{N-2}\hat{\partial}_i\hat{\partial}_j\hat{\partial}_k\hat{\partial}_l \right] \delta^{N-1}(x-y). \end{aligned} \quad (54)$$

IV. MASKAWA-NAKAJIMA'S ANALYSIS

A. Maxwell Field

Before to explore the Maskawa-Nakajima's (MN) reduction for Einstein's linearized gravity, we shall do a brief and pedagogical review of the reduction for Maxwell's electromagnetic theory for a better understanding of some of the aspects that we want to point out.

Maxwell's theory is described by the action

$$S = \left\langle \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right\rangle, \quad (55)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the Maxwell tensor and A_μ is the potential field.

Maxwell's theory is invariant under Lorentz group and the gauge group $U(1)$, hence there are redundant degrees of freedom so the theory must be reducible. A possible way to write down the reduced action \bar{S} starts with a standard decomposition of the potential field in its transverse and longitudinal parts, following the known prescription $A_i = A^T_i + \partial_i A^L$. By doing this, we can eliminate the fields that are not dynamical to rewrite the action as

$$\bar{S} = \left\langle \frac{1}{2} A^T_i \square A^T_i \right\rangle, \quad (56)$$

it can be noted the transverse part of the field A_i as the only field which carry the physical propagation.

So, the analysis of Lagrangian constraints of Maxwell's theory tells us that the temporal component of the field, in other words A_0 does not propagate, which is supported by the fact that this component appears as the Lagrange multiplier associated to the Gauss constraint in the Hamiltonian formalism and also it's canonical conjugate momentum Π_0 is a primary constraint. This allows us to focus our attention on the sub-space spanned by the $N-1$ purely spacial components of the potential field, this means A_i , whether it's in the configurations or in the phase space. The $(N-1)$ spacial part of the configuration space have $2(N-1)$ dimension because we haven't chosen a gauge yet, and we name this space as $\varepsilon_{(A_i, \dot{A}_i)}$. However, the $(N-1)$ spacial section of the phase space denoted by $\varepsilon_{(A_i, \Pi_i)}^*$ is not necessarily locally isomorph to $\varepsilon_{(A_i, \dot{A}_i)}$,

which only relies formally on whether or not we impose the Gauss constraint from the beginning.

So, starting with $\varepsilon_{(A_i, \Pi_i)}^*$ and performing the gauge fixing (i.e., the Coulomb gauge) and the Gauss constraint mean two constraints on the $(N - 1)$ spacial section of the phase space which conduce to a new and reduced one, which we shall call $\bar{\varepsilon}_{(\bar{A}_i, \bar{\Pi}_i)}^*$ with dimension $2(N - 2)$.

MN theorem[6] tells us that there exist a non one to one map from the space $\varepsilon_{(A_i, \Pi_i)}^*$ into the new space $\bar{\varepsilon}_{(\bar{A}_i, \bar{\Pi}_i)}^*$ which represents the physical degrees of freedom reduction and it must have consistency with the brackets obtained via the Dirac's reduction procedure. In this sense, we assume then a matrix representation of the projection, Ω for the $N - 1$ spacial components as follows

$$\bar{A}_k = \Omega_{kl} A_l, \quad (57)$$

$$\bar{\Pi}_k = \Omega_{kl} \Pi_l, \quad (58)$$

where we expect that (58) is redundant due to the Gauss constraint $\bar{\Pi}_i = \Pi_i$.

This projection applied to the Poisson brackets of the fields in $\varepsilon_{(A, \Pi)}$, this means $\{A_k(x), \Pi_l(y)\} = \eta_{kl} \delta^{N-1}(x - y)$ leads us to

$$\{\bar{A}_k(x), \bar{\Pi}_l(y)\} = \Omega_{km}(x) \Omega_{lm}(y) \delta^{N-1}(x - y). \quad (59)$$

We haven't said anything about the form of Ω , but since \bar{A}_k are transverse fields, we realize this transformation via the $N - 1$ transverse projector invariant under parity in the following form

$$\Omega_{ij}(x) = \Omega_{ij}(-x) \equiv \eta_{ij} + \hat{\partial}_i \hat{\partial}_j, \quad (60)$$

which satisfy

$$\Omega_{km} \Omega_{lm} = \Omega_{kl}. \quad (61)$$

Also, we can verify with this projector that the Poisson brackets that are defined in $\varepsilon_{(A_i, \Pi_i)}$ induce the expected form of Dirac's brackets [10] in $\bar{\varepsilon}_{(\bar{A}_i, \bar{\Pi}_i)}$.

The advantage of this method is that, beyond the 2+1 dimensional case, it would be a not easy bussines to find the explicit and irreducible decomposition of any tensor field of arbitrary rank and therefore the writing of Dirac's brackets in the reduced space following the MN procedure.

B. Linearized gravity

Taking in mind the last discussion, now we want to follow a similar trail to a MN reduction for Einstein's linearized gravity. We begin by making an ADM[11] decomposition in (10), exposing the transverse (T), longitudinal (L) and traceless-transverse (Tt) parts in the way $h_{ij} = h_{ij}^{Tt} + h_{ij}^T + h_{ij}^L$. After this, the non dynamical fields can be removed and the reduced action is

$$\bar{S} = \left\langle \frac{1}{2} h^{Tt}_{ij} \square h^{Tt}_{ij} \right\rangle, \quad (62)$$

which clearly shows that only the Tt part of the field propagates degrees of freedom.

From the phase space point of view, to exhibit the reduction of a phase sub-space, this means $\varepsilon_{(h_{ij}, \Pi_{ij})}^*$ in to other $\bar{\varepsilon}_{(\bar{h}_{ij}, \bar{\Pi}_{ij})}^*$ with dimension $N(N - 3)$, we must consider the N gauge fixings provided in (22) and the N Gauss constraints rewritten as $\phi_2^\mu \equiv \partial_i \Pi^{i\mu}$ with the help of the primary constraints (19). Then, we realize this reduction through traceless-transverse projector as

$$\bar{h}_{ij} = \Omega_{ijmn} h_{mn}, \quad (63)$$

$$\bar{\Pi}_{ij} = \Omega_{ijmn} \Pi_{mn}, \quad (64)$$

where the following algebraic properties must be satisfied $\Omega_{ijmn} = \Omega_{jimn} = \Omega_{ijnm}$ and

$$\Omega_{ijkl} \Omega_{mnkl} = \Omega_{ijmn}, \quad (65)$$

$$\partial_i \bar{h}_{ij} = \Omega_{ijmn} \partial_i h_{mn} = 0, \quad (66)$$

$$\bar{h}_{ii} = \Omega_{iimn} h_{mn} = 0. \quad (67)$$

An ansatz on the form of Ω_{ijmn} is

$$\Omega_{ijmn} = \frac{\alpha(N)}{2} (\Omega_{im} \Omega_{jn} + \Omega_{in} \Omega_{jm}) + \frac{\beta(N)}{2} \Omega_{ij} \Omega_{mn}, \quad (68)$$

where $\alpha(N)$ and $\beta(N)$ are unknown real coefficients and with the help of (65 - 67) we can find them as

$$\alpha(N) = 1, \quad (69)$$

$$\beta(N) = -\frac{2}{N-2}, \quad (70)$$

therefore, the projector take the form

$$\Omega_{ijkl} = \frac{1}{2} (\Omega_{ik} \Omega_{jl} + \Omega_{il} \Omega_{jk}) - \frac{1}{N-2} \Omega_{ij} \Omega_{kl}. \quad (71)$$

This projector is applied to the Poisson brackets of the fields in $\varepsilon_{(h_{ij}, \Pi_{ij})}^*$, in other words $\{h_{ij}(x), \Pi_{kl}(y)\} = \frac{(\eta_{ik} \eta_{jl} + \eta_{il} \eta_{jk})}{2} \delta^{N-1}(x - y)$, and this gives us

$$\{\bar{h}_{ij}(x), \bar{\Pi}_{kl}(y)\} = \Omega_{ijmn}(x) \Omega_{klmn}(y) \delta^{N-1}(x - y), \quad (72)$$

which is equivalent to Dirac's bracket (54).

V. CONCLUSION

The symmetries of physical systems imply the existence of conserved quantities according to Noether's theorem. This is so that when we study the action of a given system where translation invariance induces the

conservation in the lineal momentum, the conservation of energy comes from the invariance under time displacements, and so on. But all of this is accompanied by the fact that the fields that describe the theory can't be written in a unique way. Whether it is because they change under certain coordinates transformation groups or because they can simultaneously transform under functional variations as well. A typical case of this is the Maxwell's electromagnetic theory described by a tensor field of rank 1 that transforms under the Lorentz Group, which in N dimensions is denoted as $ISO(N-1, 1)$, and under the $U(1)$ group that represents the gauge invariance.

In this sense, Einstein's linearized gravity is very similar to Maxwell's case. In the perturbative regime, the symmetrical rank 2 tensor field transforms under the (local) Lorentz group and also functionally under a diffeomorphism which would be thought as a gauge transformation.

No matter which system with symmetries we're studying, there's not a unique way to approach to the true configuration of the physical fields that describe the system. From the configuration space's perspective, it is possible to make the reduction following the analysis

of Lagrangian constraints. However, in this work we've focused in the phase space and the reduction via Dirac's and MN procedures, laying down the groundwork for a possible quantization of linearized gravity.

Then, we applied Dirac's procedure to the Einstein's linearized gravity to find the correct algebra for the minimum physical fields of the theory in agreement with reference[4]. Particularly, we choose a Coulomb gauge to assure the minimal number of degrees of freedom, in analogy with the Maxwell's case.

Finally, the MN reduction has been performed conjecturing that is possible to project the unconstrained Poisson brackets on to the reduced and the algebra obtained is consistent with Dirac's procedure thanks to the use of the $N(N-1)$ traceless-transverse projector for the rank 2 tensor fields, in the same way that it's used with the $N-1$ transverse projector for the rank 1 tensor field in Maxwell's theory.

If we analyze this in perspective with all the analogies that exist between Einstein's linearized gravity and Maxwell theory, it would be interesting to explore a first order formalism for the theory, where we can confirm the reduction of the degrees of freedom, the gauge invariance, and the brackets, but this is the topic for a future work.

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